GENERAL STABILITY CRITERION FOR LAMINAR TUBE FLOW

R. S. Kuznetskii and M. K. Likht

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 10, No. 4, pp. 74–79, 1969

Initiation of turbelence is associated with disturbances of finite intensity [1]. Some attempts, as in [2], have been made to treat this region analytically. The nonuniformity of the local stability* of laminar fluid flow over the tube cross section has been established experimentally [3, 4, 1]. On this finding is based the interpretation of a number of turbulent transition phenomena, including a characteristic singularity of the relationship between the resistance coefficient of rough tubes and the Reynolds number under transient conditions [5].

Expression (1.1) introduced as a measure of stability, and the criterion q_* yield satisfactory quantitative results.

1. The degree of local stability of stationary laminar fluid flow in cylindrical (in the wide sense) tubes will be characterized by the simplex

$$\frac{\rho v \,|\nabla_{xy} v|}{|dp/dz|} = \frac{1}{2} q(x, y) \tag{1.1}$$

This simplex is analogous to the local Reynolds number introduced in [2] for plane-parallel flows, which is proportional to the transverse velocity gradient and inversely proportional to the viscosity. In (1.1), v, p, and ρ are the velocity, pressure, and density of the fluid, respectively; x and y are Cartesian coordinates in the tube cross section F; z is the coordinate along the tube axis. Simplex (1.1) constitutes the ratio of the energy variations of a fluid element for small displacements in the transverse and longitudinal directions, respectively.

Regardless of the degree of stability of laminar flow on the whole over F, there always exists a region of arbitrary smallness of q. Therefore the onset of instability should be expected for fairly large values of q. In accordance with this, we assume the existence of such a number q_* that $q \le q_*$ over the entire F is the stability condition for the flow, while the presence in F of regions with $q > q_*$ is the instability condition, i.e., the possibility of initiation of turbulence.

We reduce (1.1) to dimensionless form. For Poiseuille flow

$$\Delta_{xy} v = \frac{1}{\mu}\frac{dp}{dz} = \mathrm{const} < 0, \qquad v|_\Gamma = 0 \tag{1.2}$$

where $\mu = \rho\nu$ is dynamic viscosity and Γ is the contour of cross section F. Introducing a dimensionless velocity and dimensionless coordinates (s is the hydraulic radius of the tube)

$$u = \frac{\mu}{|dp/dz|\, s^2} v, \quad \xi = \frac{x}{s}, \quad \eta = \frac{y}{s} \tag{1.3}$$

(similarly, for the radius vector $\xi = r/s$ in a polar system of coordinates and for any linear coordinate in an arbitrary system), we have

$$\Delta_{\xi\eta} u = -1, \qquad u|_{\Gamma_{\xi\eta}} = 0 \tag{1.4}$$

Expressing q through dimensionless variables and $R = 2s\langle v\rangle/\nu$, where the angle brackets denote averaging over F, we transform (1.1) to

$$q = R\frac{u\,|\nabla_{\xi\eta} u|}{\langle u\rangle} \tag{1.5}$$

*Local stability (instability) is a negative (positive) response of the flow in a given small region to the introduced disturbances.

Simplex q is proportional to the Reynolds number and is equal to zero at the contour Γ and at points (or on lines) of extremums and at stationary-velocity points. Consequently, a point (or line) of maximum instability in which $\sup\{u|\nabla_{\xi\eta}u|\}$ takes place, will always lie beyond their circumference.

Laminar flow becomes unstable and susceptible to turbulization when $\sup\{q\} = q_*$, whence follows the critical Reynolds number

$$R_* = q_* f, \qquad f \equiv \frac{\langle u \rangle}{\sup_{F_{\xi\eta}} \{u \mid \nabla_{\xi\eta} u \mid\}} \tag{1.6}$$

which is equal to the product of q_* and a certain functional $f\{\Gamma\}$ of the shape of the cross section (i.e., of F or Γ, within similarity accuracy).

In practice, turbulization of the flow occurs at Reynolds numbers greater than R_* (delayed laminar flow conditions), since not all disturbance types are present in the flow.

2. Let us assess the influence of the shape of F on R_* and determine the value of q_* by comparing the corresponding R_* and f relations for several cross sections:

a) For a tube of circular cross section ($0 \le \zeta \le 1$) we have

$$u = {}^1/_4\ (1 - \zeta^2), \qquad q = R\, \zeta(1 - \zeta^2), \qquad f^0 = {}^3/_2 \sqrt{3} \tag{2.1}$$

the value $R_*^0 \approx 2800$ was obtained experimentally [6, 7].

b) For a plane channel ($|\eta| \le 1/2$) we have

$$u = {}^1/_8\, (1 - 4\eta^2), \qquad q = {}^3/_2 R\eta\, (1 - 4\eta^2), \qquad f = 2\sqrt{3} \tag{2.2}$$

For tubes of rectangular cross section with large side ratios (104 : 1 and 165 : 1), a value of $R_* \approx 2800$ was obtained experimentally [6-8].

For tubes and channels with cross sections more complex than a circle or an infinite strip, calculations must be made by a numerical method.

c) For a tube of square cross section ($|\xi| \le 1$, $|\eta| \le 1$), f was obtained on a computer by the net-point method with a step equal to 1/64 of the side of a square. A value of $f \approx 2.103$ was obtained.

The value $R_* \approx 2000$ was obtained experimentally [9, 6].

d) For a tube of annular cross section,

$$k \leqslant \varkappa \leqslant 1 \qquad (k = r_1 / r_2,\ \varkappa = (1 - k)\zeta = r / r_2,$$

where r_1, r_2, and r are the radius vectors of the walls and the running radius vector) we have [5-7]

$$u = \frac{1 - \varkappa^2 + \varkappa_0^2 \ln \varkappa^2}{4(1 - k)^2}, \qquad \langle u \rangle = \frac{1 + k^2 - 2\varkappa_0^2}{8(1 - k)^2}, \qquad u_{\max} = \frac{1 - \varkappa_0^2 \ln e / \varkappa_0^2}{4(1 - k)^2} \tag{2.3}$$

The quantity $\varkappa_0^2 = -(1 - k^2)/\ln k^2$ corresponds to the maximum of u. Simplex (1.1) is equal to

$$q = R \frac{(1 - \varkappa^2 + \varkappa_0^2 \ln \varkappa^2)\, |\varkappa_0^2 - \varkappa^2|}{(1 - k)(1 + k^2 - 2\varkappa_0^2)\, \varkappa} \tag{2.4}$$

whence (Fig. 1)

$$\begin{aligned} f &= (1 - k)(1 + k^2 - 2\varkappa_0^2) \frac{\varkappa}{(1 - \varkappa^2 + \varkappa_0^2 \ln \varkappa^2)(\varkappa_0^2 - \varkappa^2)} \\ &= \frac{(1 - k)(1 + k^2 - 2\varkappa_0^2)}{2} \frac{\varkappa_0^2 + \varkappa^2}{(\varkappa_0^2 - \varkappa^2)^3} \end{aligned} \tag{2.5}$$

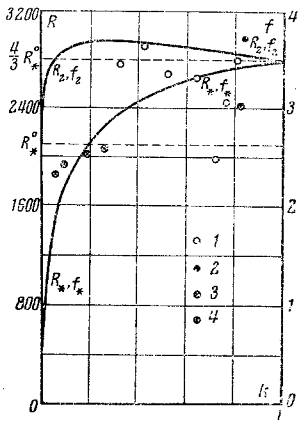

Fig. 1

Here $\varkappa^2$ is the smaller of the two roots $\varkappa_i^2(k)$ ($k < \varkappa_1 < \varkappa_0 < \varkappa_2 < 1$) (Fig. 2) of the equation

$$2(\varkappa_0^2 - \varkappa^2)^2 = (\varkappa_0^2 + \varkappa^2)(1 - \varkappa^2 + \varkappa_0^2 \ln \varkappa^2) \tag{2.6}$$

From the corresponding experiments [10–13] we shall use three, in which turbulization was observed at $R \approx 2000$, 2700, and 2640 for $k \approx 0.186$ [12], 0.514 and 0.639 [11], respectively.

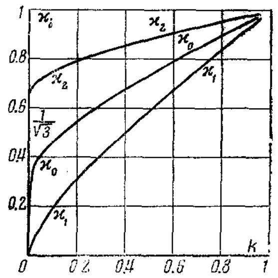

Fig. 2

A comparison of calculated values of f/f^0 with experimental values of R_*/R_*^0 is given in the table. Within the experimental and auxiliary computational uncertainty, these values are in excellent agreement. If the aforesaid effect of delayed laminar flow conditions is also taken into account, it is not difficult to come to the conclusion that q_* is universal for tubes of various cross sections.

	Tube of circular section	Plane channel	Tube of square section	Tube of annular section		
				$k = 0.186$	$k = 0.514$	$k = 0.631$
R_* / R_*^0, experiment	1	$^4/_3$ [8]	0.95 [9]	0.952 [12]	1.28 [11]	1.26 [11]
f / f^0, theory	1	$^4/_3$	0.81	1.00	1.23	1.26

Let us determine q_* with the aid of the most reliable data selected from numerous experiments performed with tubes of circular cross section [6, 7]:

$$q_* = R_*^0 / f^0 \approx 2100 : {}^3/_2 \sqrt{3} \approx 808.3 \tag{2.7}$$

The accuracy of the value obtained for q_* depends on the experimental uncertainty involved in the measurement of R_*^0.

By the same token, we get for $R_* = q_* f$ a value of 2800 for a plane channel (which is in excellent agreement with the experiment) and a value of roughly 1700 for a tube of square cross section (which is by 15% less than the experimental value).

3. Let us examine the points (lines) of maximum instability computed for tubes of various cross sections, and compare them with available data.

a) According to (2.1), the maximum of q, i.e., the circumference of maximum instability, is characterized by $\zeta = 1/\sqrt{3} \approx 0.57735$.

Experiments conducted at values of R close to R_*^0 reveal that weakest attenuation of the introduced disturbances occurs in the region $0.4 < \zeta < 0.6$ [3] and the maximum stability of the arising fluctuations is observed at $\zeta \approx 0.6$ [4]. This confirms the theoretical results.

Under turbulent conditions, the region of maximum instability of laminar flow should be characterized by a maximum turbulence level and hence by maximum turbulent viscosity. In fact, the maximum turbulent viscosity was determined experimentally for $\zeta \approx 0.6$ [14].

b) According to (2.2), the planes of maximum instability in a plane channel are characterized by $\eta = \pm {}^1/_2 \sqrt{3}$.

Thus, in a plane channel and in a tube of circular cross section, the maximum value of q occurs at like values of the relative ordinate $2|\eta|$ and the radius-vector ξ. This result can be deduced also from independent theoretical considerations [15].

c) In a tube of square cross section, the points of maximum instability lie on the symmetry axes that are parallel to the sides of the square,

$$\xi = 0, \quad \eta \approx \pm 0.625, \qquad \xi \approx \pm 0.625, \quad \eta = 0$$

The position of these points can be readily interpreted in the physical sense by comparing a square with the circle inscribed in it (paragraph a).

d) In a tube of annular cross section, q has two maxima, the larger (main) one of which is characterized by the smaller $\varkappa_1$, and the minor maximum by the large root $\varkappa_2$ of Eq. (2.6) (Fig. 2).

Inasmuch as the probability of flow turbulization increases drastically with the formation of the second of the instability zones $\varkappa \sim \varkappa_1$, $\varkappa \sim \varkappa_2$, in practice, the onset of turbulence may be expected to occur at Reynolds numbers that satisfy the inequality $R_* = R_1 < R \lessapprox R_2$, where R_i (and $f_i = R_i/q_*$) correspond to $\varkappa_i$ in accordance with (2.4) for $q = q_*$ (Fig. 1). The probability of R being close to R_2 grows when k approaches unity (i.e., when the symmetry of the flow diminishes).

In the limiting case of a small curvature of the walls $1 - k \equiv \alpha \ll 1$, we approach the results for a plane channel

$$\varkappa_0 \approx 1 - {}^1/_2\,\alpha - {}^1/_{24}\,\alpha^2, \quad \varkappa_{1,2} \approx 1 - {}^1/_2(1 \pm 1 / \sqrt{3})\,\alpha + {}^1/_4({}^{11}/_{27} \pm 1 / \sqrt{3})\alpha^2$$
$$R_{1,2} \approx 2q_*(\sqrt{3} \mp {}^1/_6\,\alpha) \tag{3.1}$$

In the opposite limiting case of $k \to 0$ (circular tube with a thin coaxial central core), R_* decreases indefinitely, while the circumference of maximum instability narrows rapidly toward the center,

$$\varkappa_0 \approx (2 \ln k)^{-1} \to 0, \quad \varkappa_1 \approx ek \to 0, \quad \varkappa_1 / \varkappa_0 \to 0, \quad R_2 \approx 2eq_*\,k \ln^2 k \to 0 \tag{3.2}$$

For the secondary local instability maximum we obtain at the limit the same results as for a circular cross section,

$$\varkappa_2 \approx \frac{1}{\sqrt{3}}\left(1 - \frac{6 - \ln 3}{4 \ln k}\right) \to \frac{1}{\sqrt{3}}\,, \; R_2 \approx R_*^0\left(1 - \frac{2 - \ln 3}{4 \ln k} - \frac{12 + 7 \ln 3 - 2 \ln^2 3}{8 \ln^2 k}\right) \to R_*^0 \tag{3.3}$$

The initiation of turbulization at $R \sim R_2 \approx R_*^0$, however, is now unlikely.

With increasing k, the quantities $\varkappa_0$, $\varkappa_1$, and R_* $(0 < R_* < {}^4/_3 R_*^0)$ increase monotonically, while $\varkappa_2$ decreases; R_2 $(R_*^0 < R_2 \leqslant 2962)$ possesses a weak maximum at $k \approx 0.30$ (Figs. 1, 2). Specifically, $R_* = R_*^0$ for $k \approx 0.18531$, and $R_2 \approx 4/3(R_*^0)$ for $k \approx 0.03323$.

Figure 1 shows also the experimentally determined values of R at which turbulization begins. Points 1, 2, 3, and 4 correspond to data in [10–13]. The agreement of (2.4) and (2.5) with experiment may be considered completely satisfactory, if we take into consideration the delaying of the laminar flow and the increasing effect of structural fabrication errors on the true R_* as the annular slit is constricted.

4. Let us examine the problem of the influence of wall roughness on the turbulization of the flow and the resistance coefficient λ of the tube.

It is natural to assume that, starting with the value of R for which the distributed roughness $\Delta = s\delta$ penetrates into the instability region $q > q_*$

$$R' = q_* \frac{\langle u \rangle}{\sup_\delta \{u \mid \nabla_{\xi\eta} u \mid\}} \tag{4.1}$$

roughness produces an appreciable change in the nature of the function λ (R) toward increasing or more intensely increasing values. Here, the subscript δ corresponds to the locus of points separated by the interval δ along the inner normal **n** (rendered nondimensional in the like manner as the coordinates) from the points on contour $\Gamma_{\xi\eta}$ (contour Γ of cross section F corresponds to the interval Δ). By the same token, the subscript 0 refers to contour $\Gamma_{\xi\eta}$.

Since the relative roughness δ is usually very small, (4.1) may be written in the form

$$R' \approx \frac{q_*}{\delta} \langle u \rangle \sup{}^{-1}\left\{\left[\left(\frac{\partial u}{\partial n}\right)_0 + \delta\left(\frac{\partial^2 u}{\partial n^2}\right)_0\right]\left[\left(\frac{\partial u}{\partial n}\right)_0 + \frac{\delta}{2}\left(\frac{\partial^2 u}{\partial n^2}\right)_0\right]\right\}$$
$$\approx \frac{q_*}{\delta} \langle u \rangle \sup{}^{-1}\left\{\left(\frac{\partial u}{\partial n}\right)_0^2\right\} \tag{4.2}$$

or

$$R_\Delta \equiv \frac{\langle v \rangle \Delta}{\nu} = \frac{R'\delta}{2} \approx \frac{q_*}{2} \langle u \rangle \sup{}^{-1}\left\{\left(\frac{\partial u}{\partial n}\right)_0^2\right\} \tag{4.3}$$

Specifically, for a tube of circular cross section

$$R' \approx \frac{q_*}{\delta(1-\delta)(2-\delta)} \approx \frac{q_*}{2\delta} = \frac{R_*^0}{3\delta\sqrt{3}} \approx \frac{404.2}{\delta}, \qquad R_\Delta \approx \frac{q_*}{4} \approx 202.1 \tag{4.4}$$

Figure 3 shows Nikuradze's experimental $\lambda(R)$ curves for rough tubes of circular cross section [5]. Points with $R = R'$ are indicated on the curves. The increase in λ, which continues until the similarity region is reached, begins at these points.

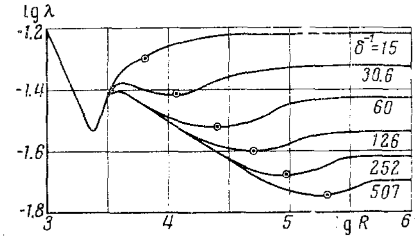

Fig. 3

This indicates that for $R_* < R < R'$, turbulent pulsations in a rough tube exhibit a tendency toward attenuation. Attenuation, however, becomes impossible as soon as constant disturbance sources begin to penetrate into the region $q > q_*$.

The "permissible" height of distributed roughness was determined experimentally for a tube of annular cross section [16, 1]. A mean value of $R_\Delta \approx 120$, and a value of $R_\Delta \approx 130$ to 140 for real negative pressure gradients were obtained.

For a plane channel, whose hydraulics is approximated by a tube with a cross section in the form of a narrow ring, it follows from (4.2) and (4.3) that

$$R' \approx \frac{q_*}{3\delta} \approx \frac{269.4}{\delta}, \qquad R_\Delta \approx \frac{q_*}{6} \approx 134.7 \tag{4.5}$$

i.e., there is good agreement with the experiment [16].

5. A certain analogy should be mentioned which exists between the stability criterion $q \le q_*$ for laminar tube flow and the Rayleigh stability criterion for Couette flow between rotating cylinders [17]. This criterion results also from a local flow energy analysis. According to the Rayleigh criterion, the flow is stable when the square of velocity circulation does not decrease anywhere as the radius vector increases,

$$\frac{d}{dr}(vr)^2 \geqslant 0 \tag{5.1}$$

and vice versa. Condition (5.1) may be written in the form

$$\frac{\rho v(-dv/dr)}{\rho v^2/r} \leqslant 1 \tag{5.2}$$

The denominator of (5.2) is the value of the pressure gradient, and the numerator is the value of the kinetic energy density of the fluid, accurate to within the minus sign.

For $dv/dr > 0$, (5.2) is fulfilled automatically, as distinct from the opposite case, where $-dv/dr = |\nabla_v|$.

Of the same sense and physical meaning is the inequality

$$\frac{1}{2} q \equiv \left| \frac{\rho v \nabla_{xy} v}{dp / dz} \right| \leqslant \frac{1}{2} q_* \tag{5.3}$$

which we have taken as the stability criterion for laminar tube flow. Here, q_* is a certain constant expressed by a natural analog of condition (5.2).

REFERENCES

1. H. Schlichting, Entstehung der Turbulenz, Springer, Berlin, 1959.
2. E. M. Khazen, "Contribution to the theory of turbulence in inhomogeneous flows," Dokl. AN SSSR, vol. 147, no. 1, 1962.
3. M. Sibulkin, "Transition from turbulent to laminar pipe flow," Phys. Fluids, vol. 5, no. 3, 1962.
4. J. C. Rotta, "Experimenteller Beitrag zur Entstehung turbulenter Strömung im Rohr," Ing.-Arch., vol. 24, no. 4, 1956.
5. L. G. Loitsyanskii, Mechanics of Liquids and Gases [in Russian], Fizmatgiz, Moscow, 1959.
6. L. Schiller, Strömung in Rohren, Akad. Verlagsgesellschaft, Leipzig, 1932.
7. Modern Developments in Fluid Dynamics, vol. 1, Clarendon Press, Oxford, 1938.
8. S. J. Davies and C. M. White, "An experimental study of the flow of water in pipes of rectangular section," Proc. Roy. Soc., ser. A, vol. 119, no. 781, p. 92, 1928.
9. L. Schiller, "Über den Strömungswiderstand von Rohren verschiedenen Ouerschnitts und Rauhigkeitsgrades," Z. angew. Math. und Mech., vol. 3, no. 1, 1923.
10. R. Winkel, "Die Wasserbewegungen in Leitungen mit Ringsspalt-Durchflussquerschnitt," Z. angew. Math. und. Mech., vol. 3, no. 4, 1923.
11. T. Lonsdale, "The flow of water in the annular space between two coaxial cylindrical pipes," Philos. Mag. and J. Sci., ser. 6, vol. 46, no. 271, p. 163, 1923.
12. F. C. Lea and A. G. Tadros, "Flow of water through a circular tube with a central core and through rectangular tubes," Philos. Mag. and J. Sci., ser. 7, vol. 11, no. 74, p. 1235, 1931.
13. A. Fage, "The influence of wall oscillations, wall rotation and entry eddies on the breakdown of laminar flow in an annular pipe," Proc. Roy. Soc., ser. A, vol. 165, no. 923, p. 520, 1938.
14. J. Laufer, "The structure of turbulence in fully developed pipe flow," NACA, TN 1174, 1954.
15. M. K. Likht and I. A. Rozhanskaya, "Classes of disturbances in hydrodynamic instability studies," Summaries of Reports to the Third All-Union Conference on Theoretical and Applied Mechanics [in Russian], Nauka, Moscow, 1968.
16. E. G. Feindt, "Untersuchungen über die Abhängigkeit des Umschlages laminar-turbulent von der Oberflächenrauhigkeit und der Druckverteilung," Jahrbuch 1956 der schiffsbautechnische Gesellschaft, vol. 50, p. 180, 1956.
17. Rayleigh, "On the dynamics of revolving fluids," Proc. Roy. Soc., ser. A, vol. 93, no. 648, p. 148, 1916.

24 February 1969

Khar'kov